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THE FIRST BETTI NUMBER OF A COMPACT ALMOST TACHIBANA SPACE

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0. Introduction

It is well known that the p-th Betti number of a compact Kählerian space is zero or even if p is odd [2]. A similar result is known for a compact Sasakian space [1], [6], [7]. In particular, the first Betti number is zero or even in a compact Sasakian space.

The purpose of this paper is to give the analogy for the first Betti number of a compact Tachibana space (=nearly Kähler space [3], = K-space [4]).

1. Preliminaries

Let M be an n-dimensional almost Hermitian space with positive definite metric $g = (g_{ji})$ and almost complex structure $J = (J_i{}^j)$, $(i, j, \cdots = 1, \cdots, n)$.

A 1-form u in M is called a covariant almost analytic form [4] if it satisfies the equation

$$\nabla_j(J_i{}^r u_r) = u_r \nabla_i J_j{}^r - J_j{}^r \nabla_r u_i ,$$

or equivalently

$$\nabla_j(J_i{}^r u_r) - \nabla_i(J_j{}^r u_r) = J_j{}^r(\nabla_r u_i - \nabla_i u_r) ,$$

where ∇ denotes the operator of covariant derivative with respect to the Riemannian connection.

An almost Hermitian space is called an almost Tachibana space (resp. a Kählerian space) if the associated 2-form $\tilde{J} = \frac{1}{2} J_{ji} dx^j \wedge dx^i$ is a Killing 2-form (resp. parallel), where we put $J_{ji} = g_{ir} J_j{}^r$ and $\{x^i\}$ is a local coordinate system of M.

Then the following theorems are known:

Theorem A [9]. *A necessary and sufficient condition for a* 1*-form u in a compact Kählerian space to be covariant analytic is that the* 1*-form u be harmonic.*

Theorem B [4]. *In a compact almost Tachibana space, a necessary and sufficient condition for a* 1*-form $u = (u_i)$ to be covariant almost analytic is that u and $\bar{u} = (J_i{}^r u_r)$ both be harmonic.*

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Throughout this paper, we shall deal with an almost Tachibana space M, that is, an almost Hermitian space satisfying

$$\nabla_j J_{ih} + \nabla_i J_{jh} = 0 . \tag{1.1}$$

We shall recall the identities in M, which are necessary for later use.
The following relations are well known [4], [8], [9]:

$$J_j{}^r R_{ri} + J_i{}^r R_{rj} = 0 , \tag{1.2}$$

$$\nabla^r \nabla_r J_{ji} = R_{jr} J_i{}^r - \tfrac{1}{2} J^{rs} R_{rsji} . \tag{1.3}$$

Next, let u be any 1-form. Then by virtue of the Ricci's identity we can obtain

$$J^{rs} \nabla_r \nabla_s u_i = -\tfrac{1}{2} J^{rs} R_{rsi}{}^t u_t . \tag{1.4}$$

If u is a harmonic 1-form, then we have

$$\nabla_j u_i - \nabla_i u_j = 0 , \qquad \nabla^r \nabla_r u_i - R_i{}^r u_r = 0 , \tag{1.5}$$

which are valid in any Riemannian space.

2. Theorems

Let us prove the following theorem.

Theorem 2.1. *In a compact almost Tachibana space M, if u is a harmonic 1-form, then $\bar{u} = (J_i{}^r u_r)$ is also so.*

Proof. Since u is a harmonic 1-form, we have

$$\nabla_i (J_j{}^r u_r) - \nabla_j (J_i{}^r u_r) = 2 u_r \nabla_i J_j{}^r + J_j{}^r \nabla_r u_i - J_i{}^r \nabla_r u_j ,$$

and therefore

$$\begin{aligned}
(u_r \nabla_i J_j{}^r) u_s \nabla^i J^{js} &+ \tfrac{1}{2} (J_i{}^r \nabla_r u_j - J_j{}^r \nabla_r u_i)(J^{is} \nabla_s u^j - J^{js} \nabla_s u^i) \\
&= (u_r \nabla_i J_j{}^r) u_s \nabla^i J^{js} + (J_i{}^r \nabla_r u_j) J^{js} \nabla_s u^j - (J_j{}^r \nabla_r u_i) J^{is} \nabla_s u^j \\
&= (u_r \nabla_i J_j{}^r) \nabla^i (J^{js} u_s) - (u_r \nabla_i J_j{}^r) J^{js} \nabla^i u_s \\
&\quad + (J_i{}^r \nabla_r u_j) J^{is} \nabla_s u^j - (J_j{}^r \nabla_r u_i) J^{is} \nabla_s u^j \\
&= (u_s \nabla_i J_j{}^s) \nabla^i (J^{jr} u_r) + (J_j{}^s \nabla_i u_s) \nabla^i (J^{jr} u_r) \\
&\quad - (J_i{}^s \nabla_s u_j) \nabla^i (J^{jr} u_r) + 3 (u_r \nabla_j J_i{}^r) J^{js} \nabla_s u^i \\
&= (\nabla^i (J^{jr} u_r)) [\nabla_i (J_j{}^s u_s) - J_i{}^s \nabla_j u_s] + 3 (u_r \nabla_j J_i{}^r) J^{jr} \nabla_r u^i \\
&= -(J^{js} u_s) \nabla^i \nabla_i (J_j{}^r u_r) + (J^{js} u_s) \nabla^i (J_i{}^r \nabla_j u_r) + 3 (u_r \nabla_j J_i{}^r) J^{js} \nabla_s u^i \\
&\quad + \tfrac{1}{2} \nabla^i \nabla_i (J^{js} u_s J_j{}^r u_r) - \nabla^i (J^{js} u_s J_i{}^r \nabla_j u_r) .
\end{aligned}$$

On the other hand, making use of (1.1), $\cdots$, (1.5) we easily see that

$$\nabla^r(J_r{}^s\nabla_s u_i) = \nabla^r\nabla_r(J_i{}^s u_s) , \qquad (u_r\nabla_i J_j{}^r)J^{is}\nabla_s u^j = 0 .$$

Hence, by Green's theorem and the obvious fact that $\nabla^r(J_r{}^s u_s) = 0$, the theorem is proved.

As a corollary of this theorem, we obtain

Theorem 2.2. *The first Betti number of a compact almost Tachibana space is zero or even.*

By virtue of Theorem B and Theorem 2.1, we get

Theorem 2.3. *In a compact almost Tachibana space, a necessary and sufficient condition for a 1-form u to be covariant almost analytic is that u be harmonic.*

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